MATH 245 S18, Exam 2 Solutions

1. Carefully define the following terms: Proof by Contradiction theorem, Nonconstructive Existence Proof theorem, Existence and Uniqueness Proof theorem, Fibonacci numbers.

   The Proof by Contradiction theorem states that for any propositions p, q, if $(p \wedge \neg q) \equiv F$, then $p \rightarrow q$ is true. The Nonconstructive Existence Proof theorem states that if $(\forall x \in D, \neg P(x)) \equiv F$, then $\exists x \in D, P(x)$ is true. The Existence and Uniqueness Proof theorem states that to prove there is exactly one $x \in D$ for which $P(x)$ holds, we prove both (a) $\exists x \in D, P(x)$ and (b) $\forall x, y \in D, (P(x) \wedge P(y)) \rightarrow (x = y)$. The Fibonacci numbers are a recurrence given by $F_0 = 0, F_1 = 1, F_k = F_{k-1} + F_{k-2}$ (for $k \geq 2$).

2. Carefully define the following terms: Proof by (basic) Induction, Proof by Reindexed Induction, big Omega (Ω), big Theta (Θ).

   To prove $\forall x \in \mathbb{N}, P(x)$ by (basic) induction, we must (a) prove that $P(1)$ is true, and (b) prove $\forall x \in \mathbb{N}, P(x) \rightarrow P(x+1)$. To prove $\forall x \in \mathbb{N}, P(x)$ by reindexed induction, we must (a) prove that $P(1)$ is true, and (b) prove $\forall x \in \mathbb{N}$ with $x \geq 2$, $P(x-1) \rightarrow P(x)$. Given sequences a_n, b_n, we say that $a_n = \Omega(b_n)$ if $\exists n_0 \in \mathbb{N}, \exists M \in \mathbb{R}, \forall n \geq n_0, M|a_n| \geq |b_n|$. Given sequences a_n, b_n, we say that $a_n = \Theta(b_n)$ if $a_n = O(b_n)$ and $a_n = \Omega(b_n)$.

3. Suppose that an algorithm has runtime specified by recurrence relation $T_n = 4T_{n/2} + n^2$. Determine what, if anything, the Master Theorem tells us.

   In the notation of the Master Theorem, $a = 4, b = 2, c_n = n^2$. We calculate $d = \log_2 4 = 2$. Since $c_n = n^2 = \Theta(n^2) = \Theta(n^d)$, the "middle c_n" case applies, telling us that $T_n = \Theta(n^2 \log n)$.

4. Let $x \in \mathbb{R}$. Prove that $\lceil x \rceil$ is unique; that is, prove that there is at most one $n \in \mathbb{Z}$ with $n - 1 < x \leq n$.

   Let $n, n' \in \mathbb{Z}$ with $n - 1 < x \leq n$ and $n' - 1 < x \leq n'$. We have $n - 1 < x \leq n'$ so $n - 1 < n'$ and $n < n' + 1$. But also $n' - 1 < x \leq n$ so $n' - 1 < n$. Combining the two inequalities, we get $n' - 1 < n < n' + 1$. Applying Theorem 1.12 (since $n, n' \in \mathbb{Z}$), we conclude that $n = n'$.

5. Let $x \in \mathbb{R}$. Prove that $\lceil x \rceil$ exists; that is, prove that there is at least one $n \in \mathbb{Z}$ with $n - 1 < x \leq n$.

   We will use minimum element induction. Define $S = \{m \in \mathbb{Z} : m \geq x\}$, a nonempty set of integers with x as a lower bound. Hence S has some minimum element n. $x \leq n$ because $n \in S$. We have two cases: if $n - 1 < x$, we are done. If instead $n - 1 \geq x$, then $n - 1$ is an integer, and $\geq x$, so $n - 1 \in S$. But then n wasn't the minimum element, a contradiction. Hence $n - 1 < x \leq n$.

6. Let $n \in \mathbb{Z}$. Prove that $\frac{(n-1)n(n+1)}{3} \in \mathbb{Z}$.

We apply the Division Algorithm to get integers q, r with $n = 3q + r$ and $0 \le r < 3$. There are three cases:
Case $r = 0$: $\frac{(n-1)n(n+1)}{3} = \frac{(n-1)3q(n+1)}{3} = (n-1)q(n+1)$.
Case $r = 1$: $\frac{(n-1)n(n+1)}{3} = \frac{(3q+1-1)n(n+1)}{3} = qn(n+1)$.
Case $r = 2$: $\frac{(n-1)n(n+1)}{3} = \frac{(n-1)n(3q+2+1)}{3} = (n-1)n(q+1)$.
In all three cases, $\frac{(n-1)n(n+1)}{3}$ is an integer.

7. Solve the recurrence given by $a_0 = 0, a_1 = 6$, $a_n = a_{n-1} + 2a_{n-2}$ (for $n \ge 2$).

The characteristic polynomial is $r^2 = r+2$, rearranged as $0 = r^2-r-2 = (r-2)(r+1)$. This has roots $2, -1$, so the general solution is $a_n = A2^n+B(-1)^n$. We now use the initial conditions as $0 = a_0 = A2^0+B(-1)^0 = A+B$ and $6 = a_1 = A2^1+B(-1)^1 = 2A-B$. Solving the linear system $\{0 = A+B, 6 = 2A-B\}$ we get $A = 2, B = -2$. Hence the specific solution is $a_n = 2 \cdot 2^n + (-2) \cdot (-1)^n = 2^{n+1} + 2(-1)^{n+1}$.

8. Let $r \in \mathbb{R}$. Use induction to prove that for all $n \in \mathbb{N}_0$, $(1-r)\sum_{i=0}^{n} r^i = 1 - r^{n+1}$.

Proof by (shifted) induction on n.
Base case, $n = 0$: LHS is $(1-r)r^0 = 1-r$. RHS is $1 - r^1 = 1 - r$.
Inductive case: Assume that $(1-r)\sum_{i=0}^{n} r^i = 1-r^{n+1}$. We now add $(1-r)r^{n+1}$ to both sides, getting $(1-r)\sum_{i=0}^{n+1} r^i = (1-r)r^{n+1} + (1-r)\sum_{i=0}^{n} r^i = (1-r)r^{n+1} + 1 - r^{n+1} = r^{n+1} - r^{n+2} + 1 - r^{n+1} = 1 - r^{n+2}$. Hence $(1-r)\sum_{i=0}^{n+1} r^i = 1 - r^{n+2}$.

9. Without using the Classification Theorem, prove that $a_n = O(4^n)$, for $a_n = 3^n$. Hint: induction.

We will first use induction to prove that $\forall n \in \mathbb{N}, 3^n \le 4^n$. Base case, $n = 1$: $3^1 = 3 < 4 = 4^1$. Assume now that $3^n \le 4^n$. Multiply both sides by 3 to get $3^{n+1} = 3 \cdot 3^n \le 3 \cdot 4^n$. Now, since $3 < 4$, we multiply both sides by 4^n to get $3 \cdot 4^n < 4 \cdot 4^n$. Combining, we conclude that $3^{n+1} \le 4^{n+1}$.

Lastly, set $n_0 = 1, M = 1$ and let $n \ge n_0$ be arbitrary. $|a_n| = 3^n \le 4^n = 1|4^n| = M|4^n|$.

10. Without using the Classification Theorem, prove that $a_n \ne O(2^n)$, for $a_n = 3^n$.

Let $n_0 \in \mathbb{N}$, $M \in \mathbb{R}$ be arbitrary. Set $n = \max(n_0, 1 + \lceil \frac{\ln M}{\ln 3 - \ln 2} \rceil)$. We have $n \ge n_0$, and also $n > \frac{\ln M}{\ln 3 - \ln 2} = \frac{\ln M}{\ln(3/2)} = \log_{3/2}(M)$. We exponentiate both sides to base 3/2 to get $(\frac{3}{2})^n > (\frac{3}{2})^{\log_{3/2}(M)} = M$ or $\frac{3^n}{2^n} > M$. Hence $|a_n| = 3^n > M2^n = M|2^n|$.